Morse functions and cohomology of homogeneous spaces

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Morse functions are useful tool in revealing the geometric formation of its domain manifolds M. They define the handle decompositions of M from which the additive homologies $H_*(M)$ may be constructed. In these lectures two further questions were emphasized.

(1) How to find a Morse function on a given manifold?

(2) From Morse functions can one derive the multiplicative cohomology rather than the additive homology?

It is not our intention here to make detailed studies of these question. Instead, we will illustrate by examples solutions to them for some classical manifolds as homogeneous spaces.

I am very grateful to Piotr Pragacz for the opportunity to speak of the wonder that I have experienced with Morse functions, and for his hospitality during my stay in Warsaw. Thanks are also due to Dr. Marek Szyjewski for taking the lecture notes from which the present article was initiated, and to Dr. M. Borodzik for many improvements on the earlier version of the note.

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1 Computing homology: a classical method

There are many ways to introduce Morse Theory. However, I would like to present it in the effective computation of homology (cohomology) of manifolds.

Homology (cohomology) theory is a bridge between geometry and algebra in the sense that it assigns to a manifold M a graded abelian group $H_*(M)$ (graded ring $H^*(M)$), assigns to a map $f : M \to N$ between manifolds the induced homomorphism

$$f_* : H_*(M) \to H_*(N) \text{ (resp. } f^* : H^*(N) \to H^*(M)).$$

During the past century this idea has been widely applied to translate geometric problems concerning manifolds and maps between them to problems about groups (or rings) and homomorphisms, so that by solving the latter in the well-developed framework of algebra, one obtains solutions to the problems initiated from geometry.

The first problem one encounters when working with homology theory is the following one.

Problem 1. *Given a manifold M, compute $H_*(M)$ (as a graded abelian group) and $H^*(M)$ (as a graded ring).*

We begin by recalling a classical method to approach the additive homology of manifolds.

1–1. Homology of a cell complex

The simplest geometric object in dimension n, $n \geq 0$, is the unit ball $D^n = \{x \in \mathbb{R}^n \mid \|x\|^2 \leq 1\}$ in the Euclidean n-space $\mathbb{R}^n = \{x = (x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$, which will be called the *n-dimensional disk* (or *cell*) . Its boundary presents us the simplest closed $(n-1)$ dimensional manifold, the $(n-1)$ *sphere*: $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n \mid \|x\|^2 = 1\}$.

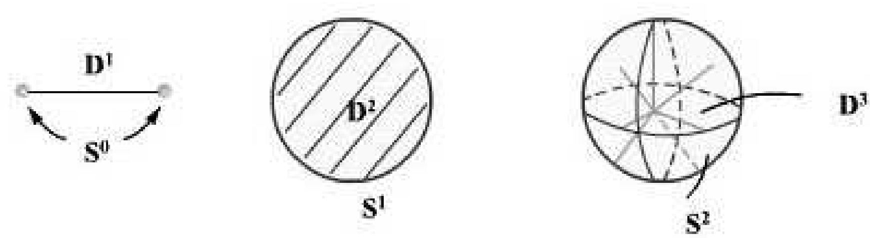

Figure 1. Cells of small dimension n=1,2,3

Let $f : S^{r-1} \to X$ be a continuous map from S^{r-1} to a topological space X. Using f we define

(1) an adjunction space

$$X_f = X \cup_f D^r = X \sqcup D^r / y \in S^{r-1} \sim f(y) \in X,$$

called *the space obtained from* X *by attaching an n-cell using* f.

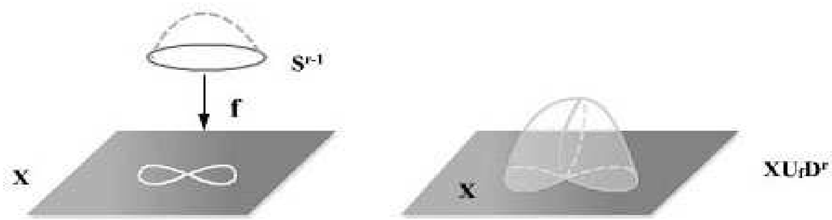

Figure2. Attaching a cell

(2) a homology class $f_*[S^{r-1}] \in H_{r-1}(X;\mathbb{Z})$ which generates a cyclic subgroup of $H_{r-1}(X;\mathbb{Z})$: $a_f =< f_*[S^{r-1}] >\subset H_{r-1}(X;\mathbb{Z})$.

We observe that the integral homology of the new space $X \cup_f D^r$ can be computed in terms of $H_*(X;\mathbb{Z})$ and its subgroup a_f.

Theorem 1. *Let* $X_f = X \cup_f D^r$. *Then the inclusion* $i : X \to X_f$

1) induces isomorphisms $H_k(X;\mathbb{Z}) \to H_k(X_f;\mathbb{Z})$ *for all* $k \neq r, r-1$;
2) fits into the short exact sequences

$$0 \to a_f \to H_{r-1}(X;\mathbb{Z}) \xrightarrow{i_*} H_{r-1}(X_f;\mathbb{Z}) \to 0$$
$$0 \to H_r(X;\mathbb{Z}) \xrightarrow{i_*} H_r(X_f;\mathbb{Z}) \to \left\{ \begin{array}{l} 0 \text{ if } |a_f| = \infty \\ \mathbb{Z} \to 0 \text{ if } |a_f| < \infty. \end{array} \right.$$

Proof. Substituting in the homology exact sequence of the pair (X_f, X)

$$H_k(X_f, X;\mathbb{Z}) = \left\{ \begin{array}{l} 0 \text{ if } k \neq r; \\ \mathbb{Z} \text{ if } k = r \end{array} \right.$$

(note that the boundary operator maps the generator of $H_r(X_f, X;\mathbb{Z}) = \mathbb{Z}$ to $f_*[S^{r-1}]$), one obtains (1) and (2) of the Theorem.□

Definition 1.1. *Let* X *be a topological space. A cell-decomposition of* X *is a sequence of subspaces* $X_0 \subset X_1 \subset \cdots \subset X_{m-1} \subset X_m = X$ *so that*
a) X_0 *consists of finite many points* $X_0 = \{p_1, \cdots, p_l\}$; *and*

b) $X_k = X_{k-1} \cup_{f_i} D^{r_k}$*, where* $f_i : \partial D^{r_k} = S^{r_k - 1} \to X_{k-1}$ *is a continuous map.*

Moreover, X *is called a cell complex if a cell-decomposition of* X *exists.*

Two comments are ready for the notion of cell-complex X.

> (1) It can be build up using the simplest geometric objects D^n, $n = 1, 2, \cdots$ by repeated applying the same construction as "attaching cell";
>
> (2) Its homology can be computed by repeated applications of the single algorithm (i.e. Theorem 1).

The concept of cell-complex was initiated by Ehresmann in 1933-1934. Suggested by the classical work of H. Schubert in algebraic geometry in 1879 [Sch], Ehresmann found a cell decomposition for the complex Grassmannian manifolds from which the homology of these manifolds were computed [Eh]. The cells involved are currently known as *Schubert cells* (*varieties*) [MS].

In 1944, Whitehead [Wh] described a cell decomposition for the real Stiefel manifolds (including all real orthogonal groups) in order to compute the homotopy groups of these manifolds, where the cells were called *the normal cells* by Steenrod [St] or *Schubert cells* by Dieudonné [D, p.226]. In terms of this cell decompositions the homologies of these manifolds were computed by C. Miller in 1951 [M]. We refer the reader to Steenrod [St] for the corresponding computation for complex and quaternionic Stiefel manifolds.

Historically, finding a cell decomposition of a manifold was a classical approach to computing its homology. It should be noted that it is generally a difficult and tedious task to find (or to describe) a cell-decomposition for a given manifold. We are looking for simpler methods.

1–2. Attaching handles (Construction in manifolds)

"Attaching cells" is a geometric procedure to construct topological spaces by using the elementary geometric objects D^r, $r \geq 0$. The corresponding construction in manifolds are known as "attaching handles"or more intuitively, "*attaching thickened cells*".

Let M be an n-manifold with boundary $N = \partial M$, and let $f : S^{r-1} \to N$ be a smooth embedding of an $(r-1)$-sphere whose tubular neighborhood in N is trivial: $T(S^{r-1}) = S^{r-1} \times D^{n-r}$. Of course, as in the previous section, one may form a new topological space $M_f = M \cup_f D^r$ by attaching an r-cell to M by using f. However, the space M_f is in general not a manifold!

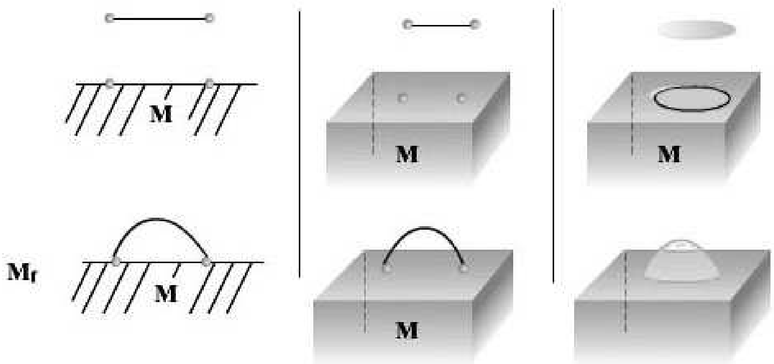

Figure3: Attaching cells using embedding
f: $S^{r-1} \to \partial M \subset M$

Nevertheless, one may construct a new manifold M' which contains the space M_f as a "*strong deformation retract*" by the procedure below.

Step 1. To match the dimension of M, thicken the r-disc D^r by taking product with D^{n-r}

$$D^r \times 0 \subset D^r \times D^{n-r} \text{ (a thickened } r\text{-disc)}$$

and note that $\partial(D^r \times D^{n-r}) = S^{r-1} \times D^{n-r} \cup D^r \times S^{n-r-1}$.

Step 2. Choose a diffeomorphism

$$S^{r-1} \times D^{n-r} (\subset D^r \times D^{n-r}) \xrightarrow{\varphi} T(S^r) \subset M$$

that extends f in the sense that $\varphi \mid S^{r-1} \times \{0\} = f$;

Step 3. Gluing $D^r \times D^{n-r}$ to M by using φ to obtain $M' = M \cup_\varphi D^r \times D^{n-r}$.

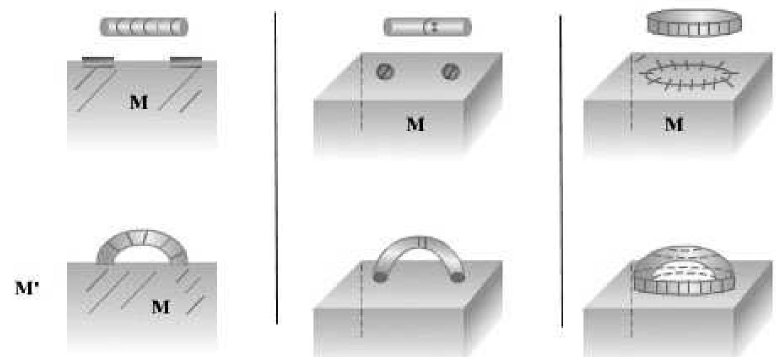

Figure 4. Attaching handles (thickened cells): the resultting space M' is a manifold.

Step 4. Smoothing the angles [M_3].

Definition 1.2. *M' is called the manifold obtained from M by adding a thickened $r-$cell with core M_f.*

Remark. The homotopy type (hence the homology) of M' depends on the homotopy class $[f] \in \pi_{r-1}(M)$ of f.

The diffeomorphism type of M' depends on the isotopy class of the embedding f (with trivial normal bundle), and a choice of $\varphi \in \pi_r(SO(n-r))$.

Inside $M' = M \cup_\varphi D^r \times D^{n-r}$ one finds the submanifold $M \subset M'$ as well as the subspace $M_f = M \cup_f D^r \times \{0\} \subset M' = M \cup_\varphi D^r \times D^{n-r}$ in which the inclusion $j : M_f \to M'$ is a homotopy equivalence. In particular, j induces isomorphism in every dimension

$$H_k(M_f, \mathbb{Z}) \to H_k(M'; \mathbb{Z}),\ k \geq 0.$$

Consequently, the integral cohomology of the new manifold M' can be expressed in terms of that of M together with the class $f_*[S^{r-1}] \in H_{r-1}(M;\mathbb{Z})$ by Theorem 1.

Corollary. *Let M' be the manifold obtained from M by adding a thickened $r-$cell with core M_f. Then the inclusion $i : M \to M'$*

1) induces isomorphisms $H_k(M;\mathbb{Z}) \to H_k(M';\mathbb{Z})$ for all $k \neq r, r-1$;

2) fits into the short exact sequences

$$0 \to a_f \to H_{r-1}(M;\mathbb{Z}) \to H_{r-1}(M';\mathbb{Z}) \to 0$$
$$0 \to H_r(M;\mathbb{Z}) \to H_r(M';\mathbb{Z}) \to \left\{ \begin{array}{c} 0 \text{ if } |a_f| = \infty \\ \mathbb{Z} \to 0 \text{ if } |a_f| < \infty. \end{array} \right.$$

Definition 1.3. *Let M be a smooth closed n-manifold (with or without boundary). A handle decomposition of M is a filtration of submanifolds $M_1 \subset M_2 \subset \cdots \subset M_{m-1} \subset M_m = M$ so that*

(1) $M_1 = D^n$;

(2) M_{k+1} is a manifold obtained from M_k by attaching a thickened r_k-cell, $r_k \leq n$.

If M is endowed with a handle decomposition, its homology can be computed by repeated applications of the corollary

$$H_*(M_1) \mapsto H_*(M_2) \mapsto \cdots \mapsto H_*(M).$$

Now, Problem 1 can be stated in geometric terms.

Problem 2. *Let M be a smooth manifold.*

(1) Does M admits a handle decomposition?

(2) If yes, find one.

2 Elements of Morse Theory

Using Morse function we prove, in this section, the following result which answers (1) of Problem 2 affirmatively.

Theorem 2. *Any closed smooth manifold admits a handle decomposition.*

2–1. Study manifolds by using functions: the idea

Let M be a smooth closed manifold of dimension n and let $f : M \to \mathbb{R}$ be a non-constant smooth function on M. Put

$$a = \min\{f(x) \mid x \in M\}, \; b = \max\{f(x) \mid x \in M\}.$$

Then f is actually a map onto the interval $[a, b]$.

Intuitively, f assigns to each point $x \in M$ a *height* $f(x) \in [a, b]$. For a $c \in (a, b)$, those points on M with the same height c (i.e. $L_c = f^{-1}(c)$) form *the level surface* of f at level c. It cuts the whole manifold into two parts $M = M_c^- \cup M_c^+$ with

$$M_c^- = \{x \in M \mid f(x) \leq c\} \text{ (the part below } L_c)$$
$$M_c^+ = \{x \in M \mid f(x) \geq c\} \text{ (the part above } L_c)$$

and with $L_c = M_c^- \cap M_c^+$.

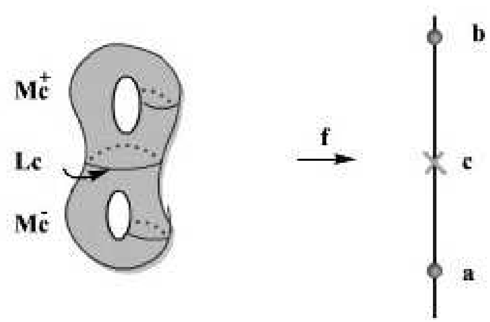

Figure 5. The level surface Lc cuts M into two parts.

In general, given a sequence of real numbers $a = c_1 < \cdots < c_m = b$, the $m - 2$ level surfaces L_{c_i}, $2 \leq i \leq m - 1$, defines a filtration on M

$$M_1 \subset M_2 \subset \cdots \subset M_{m-1} \subset M_m = M,$$

with $M_i = M_{c_i}^-$.

Our aim is to understand the geometric construction of M (rather than the functions on M). Naturally, one expects to find a good function f as well as suitable reals $a = c_1 < c_2 < \cdots < c_m = b$ so that

(1) each M_i is a smooth manifold with boundary L_{c_i};

(2) the change in topology between each adjoining pair $M_k \subset M_{k+1}$ is as simple as possible.

If this can be done, we may arrive at a global picture of the construction of M.

Among all smooth functions on M, Morse functions are the most suitable for this purpose.

2–2. Morse functions

Let $f : M \to \mathbb{R}$ be a smooth function on a n-dimensional manifold M and let $p \in M$ be a point. In a local coordinates $(x_1, \cdots, x_n)$ centered at p (i.e. a Euclidean neighborhood around p) the Taylor expansion of f near p reads

$$f(x_1, \cdots, x_n) = a + \sum_{1 \le i \le n} b_i x_i + \sum_{1 \le i,j \le n} c_{ij} x_i x_j + o(\| x \|^3),$$

in which

$$a = f(0); \qquad b_i = \frac{\partial f}{\partial x_i}(0),\ 1 \le i \le n; \text{ and}$$
$$c_{ij} = \frac{1}{2} \frac{\partial^2 f}{\partial x_j \partial x_i}(0),\ 1 \le i, j \le n.$$

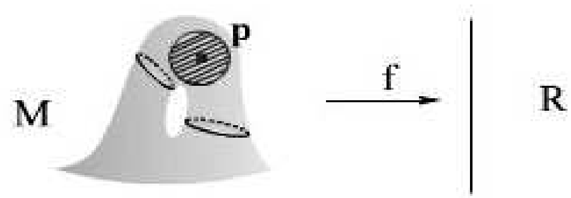

Figure 6 a Eudidean neighborhood around a point p∈M

Let T_pM be the tangent space of M at p. The $n \times n$ symmetric matrix,

$$H_p(f) = (c_{ij}) : T_pM \times T_pM \to \mathbb{R} \text{ (resp. } T_pM \to T_pM)$$

called the *Hessian form* (resp. Hessian operator) of f at p, can be brought into diagonal form by changing the linear basis $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ of T_pM

$$H_p(f) = (c_{ij}) \sim 0_s \oplus (-I_r) \oplus (I_t),\ s + r + t = n.$$

Definition 2.1. *$p \in M$ is called a critical point of f if in a local coordinates at p, $b_i = 0$ for all $1 \leq i \leq n$. Write Σ_f for the set of all critical points of f.*

A critical point $p \in \Sigma_f$ is called non-degenerate if the form $H_p(f)$ is non-degenerate. In this case the number r is called the index of p (as a non-degenerate critical point of f), and will be denoted by $r = Ind(p)$.

f is said to be a Morse function on M if all its critical points are non-degenerate.

The three items "*critical point*", "*non-degenerate critical point*" as well as the "*index*" of a nondegenerate critical point specified in the above are clearly independent of the choice of local coordinates centered at p. Two useful properties of a Morse function are given in the next two lemmas.

Lemma 2.1. *If M is closed and if f is a Morse function on M, then Σ_f is a finite set.*

Proof. The set Σ_f admits an intrinsic description without referring to local coordinate systems.

The tangent map $Tf : TM \to \mathbb{R}$ of f gives rise to a cross section $\sigma_f : M \to T^*M$ for the cotangent bundle $\pi : T^*M \to M$. Let $\sigma : M \to T^*M$ be the zero section of π. Then $\Sigma_f = \sigma_f^{-1}[\sigma(M)]$. f is a Morse function is equivalent to the statement that the two embeddings $\sigma_f, \sigma : M \to T^*M$ have transverse intersection.□

Lemma 2.2 (**Morse Lemma**, cf. [H; p.146]). *If $p \in M$ is a non-degenerate critical point of f with index r, there exist local coordinates $(x_1, \cdots, x_n)$ centered at p so that*

$$f(x_1, \cdots, x_n) = f(0) - \sum_{1 \leq i \leq r} x_i^2 + \sum_{r < i \leq n} x_i^2$$

(i.e. the standard nondegenerate quadratic function of index r).

Proof. By a linear coordinate change we may assume that

$$\left(\frac{\partial^2 f}{\partial x_j \partial x_i}(0)\right) = (-I_r) \oplus (I_{n-r}).$$

Applying the fundamental Theorem of calculus twice yields the expansion

$$\text{(A)} \qquad f(x_1, \cdots, x_n) = f(0) + \sum_{1 \leq i,j \leq n} x_i x_j b_{ij}(x)$$

in which

$$b_{ij}(x) = \int_0^1 \int_0^1 \frac{\partial^2 \overline{f}}{\partial x_j \partial x_i}(stx_1, \cdots, stx_n) dtds.$$

The family of matrix $B(x) = (b_{ij}(x))$, $x \in U$, may be considered as a smooth map

$$B : U \to \mathbb{R}^{\frac{n(n+1)}{2}} (=\text{the vector space of all } n \times n \text{ symmetric matrices}).$$

with $B(0) = (-I_r) \oplus (I_{n-r})$, where $U \subset M$ is the Euclidean neighborhood centered at p. It follows that

"*there is a smooth map* $P : U \to GL(n)$ *so that in some neighborhood* V *of* $0 \in U$,

$$B(x) = P(x)\{(-I_r) \oplus (I_{n-r})\}P(x)^\tau \text{ and } P(0) = I_n\text{''}.$$

With this we infer from (A) that, for $x = (x_1, \cdots, x_n) \in V$

$$f(x) = f(0) + xB(x)x^\tau = f(0) + xP(x)\{(-I_r) \oplus (I_{n-r})\}P(x)^\tau x^\tau.$$

It implies that if one makes the coordinate change

$$(y_1, \cdots, y_n) = (x_1, \cdots, x_n)P(x)$$

on a neighborhood of $0 \in U$ then one gets

$$f(y_1, \cdots, y_n) = f(0) - \sum_{1 \le i \le r} y_i^2 + \sum_{r < i \le n} y_i^2 . \square$$

2–3. Geometry of gradient flow lines

The first set of information we can derive directly from a Morse function $f : M \to \mathbb{R}$ consists of

(1) the set Σ_f of critical points of f;

(2) the index function $Ind : \Sigma_f \to \mathbb{Z}$.

Equip M with a Riemannian metric so that the gradient field of f

$$v = grad(f) : M \to TM,$$

is defined. One of the very first thing that one learns from the theory of ordinary differential equations is that, for each $x \in M$, there exists a unique smooth curve $\varphi_x : \mathbb{R} \to M$ subject to the following constraints

(1) the initial condition: $\varphi_x(0) = x$;

(2) the ordinary differential equation: $\frac{d\varphi_x(t)}{dt} = v(\varphi_x(t))$;

(3) φ_x varies smoothly with respect to $x \in M$ in the sense that

"the map $\varphi : M \times \mathbb{R} \to M$ by $(x,t) \to \varphi_x(t)$ is smooth and, for every $t \in \mathbb{R}$, the restricted function $\varphi : M \times \{t\} \to M$ is a diffeomorphism."

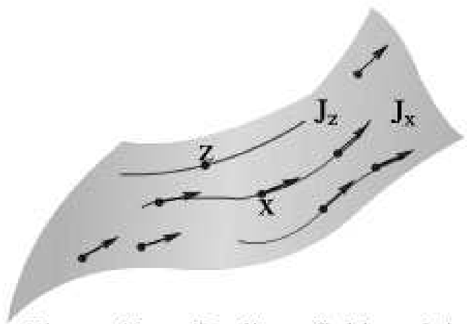

Figure 7 Gradient field and its flow lines

Definition 2.2. *For $x \in M$ let $J_x = \mathrm{Im}\ \varphi_x \subset M$, and call it the gradient flow line of f through x.*

An alternative description for J_x is the following. It is the image of the parameterized curve $\varphi(t)$ in M that satisfies

1) passing through x at the time $t = 0$;

2) at any point $y \in J_x$, the tangent vector $\frac{d\varphi}{dt}$ to J_x at y agrees with the value of v at y.

We build up the geometric picture of flow lines in the result below.

Lemma 2.3 (Geometry of gradient flow lines).

(1) $x \in \Sigma_f \Leftrightarrow J_x$ consists of a point;

(2) $\forall x, y \in M$ we have either $J_x = J_y$ or $J_x \cap J_y = \emptyset$;

(3) if $x \notin \Sigma_f$, then J_x meets level surfaces of f transversely; and f is strictly increasing along the directed curve J_x;

(4) if $x \notin \Sigma_f$, the two limits $\lim_{t \to \pm\infty} \varphi_x(t)$ exist and belong to Σ_f.

Figure 8

Geometry of gradient flow lines: traveling between critical points and perpendicular to regular level surfaces

Proof. (2) comes from the fact that $\varphi_{\varphi_x(t)}(s) = \varphi_x(t+s)$.
(3) is verified by

$$\frac{df\varphi_x(t)}{dt} = <\text{grad}f, \frac{d\varphi_x(t)}{dt}> = |\text{grad}f\ |^2 > 0.$$

Since the function $f\varphi_x(t)$ is bounded $a \leq f\varphi_x(t) \leq b$ and is monotone in t, the limits $\lim\limits_{t\to\pm\infty} f\varphi_x(t)$ exist. It follows from (3) that $\lim\limits_{t\to\pm\infty} |\text{grad}_{\varphi_x(t)}f\ |^2 = 0$. This shows (4).□

The most important notion subordinate to flow lines is:
Definition 2.3. *For a $p \in \Sigma_f$ we write*

$$S(p) = \bigcup_{\lim\limits_{t\to+\infty}\varphi_x(t)=p} J_x \cup \{p\};\ \ T(p) = \bigcup_{\lim\limits_{t\to-\infty}\varphi_x(t)=p} J_x \cup \{p\}.$$

These will be called respectively the descending cell and the ascending cell of f at the critical point p.

The term "*cell*" appearing in Definition 2.3 is justified by the next result.
Lemma 2.4. *If $p \in \Sigma_f$ with Ind$(p) = r$, then $(S(p), p) \cong (R^r, 0)$, $(T(p), p) \cong (R^{n-r}, 0)$, and both meet transversely at p.*

Proof. Let $(\mathbb{R}^n, 0) \subset (M, p)$ be an Euclidean neighborhood centered at p so that

$$f(x, y) = f(0) - |\ x\ |^2 + |\ y\ |^2 (\text{cf. Lemma 2.2}),$$

where $(x, y) \in \mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^{n-r}$. We first examine $S(p) \cap \mathbb{R}^n$ and $T(p) \cap \mathbb{R}^n$.

On $\mathbb{R}^n$ the gradient field of f is easily seen to be grad$f = (-2x, 2y)$. The flow line J_{x_0} through a point $x_0 = (a, b) \in \mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^{n-r}$ is

$$\varphi_{x_0}(t) = (ae^{-2t}, be^{2t}), t \in \mathbb{R}.$$

Now one sees that

$$x_0 \in S(p) \cap \mathbb{R}^n \Longleftrightarrow \lim_{t \to +\infty} \varphi_{x_0}(t) = 0(p) \Longleftrightarrow b = 0;$$
$$x_0 \in T(p) \cap \mathbb{R}^n \Longleftrightarrow \lim_{t \to -\infty} \varphi_{x_0}(t) = 0(p) \Longleftrightarrow a = 0.$$

It follows that

$$\text{(B)} \qquad S(p) \cap \mathbb{R}^n = \mathbb{R}^r \oplus \{0\} \subset \mathbb{R}^n; \quad T(p) \cap \mathbb{R}^n = \{0\} \oplus \mathbb{R}^{n-r} \subset \mathbb{R}^n$$

and both sets meet transversely at $0 = p$.

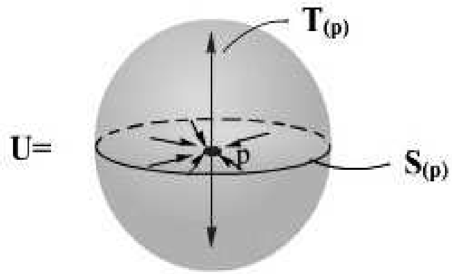

Figure 9 $\quad$ S(p)=U flow lines converge to p

T(p)=U flow lines departing from p

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and put

$$S_- = S(p) \cap S^{n-1} \text{ (resp. } S_+ = T(p) \cap S^{n-1}).$$

Then (B) implies that $S_- \cong S^{r-1}$ (resp. $S_+ \cong S^{n-r-1}$). Furthermore, (2) of Lemma 2.3 implies that, for any $x \in S(p)$, $J_x = J_v$ for some unique $v \in S_-$ because of $\varphi_x(t) \in S(p) \cap \mathbb{R}^n$ for sufficient large t with $\lim_{t \to +\infty} \varphi_x(t) = p$. Therefore

$$S(p) = \bigcup_{v \in S_-} J_v \cup \{p\} \text{ (resp. } T(p) = \bigcup_{v \in S_+} J_v \cup \{p\}).$$

That is, $S(p)$ (resp. $T(p)$) is an open cone over S_- (resp. S_+) with vertex p.□

Summarizing, at a critical point $p \in \Sigma_f$,

(1) the flow lines that grow to p (as $t \to \infty$) form an open cell of dimension $Ind(p) = r$ centered at p which lies below the critical level $L_{f(p)}$;

(2) those flow lines that grow out of from p (as $t \to \infty$) form an open cell of dimension $Ind(p) = n - r$ centered at p which lies above the critical level $L_{f(p)}$.

2–4. Handle decomposition of a manifold

Our proof of Theorem 2 implies that the set of descending cells $\{S(p) \subset M \mid p \in \Sigma_f\}$ of a Morse function on M endows M with the structure of a cell complex.

Proof of Theorem 2. Let $f : M \to [a, b]$ be a Morse function on a closed manifold M with critical set Σ_f and index function $Ind : \Sigma_f \to \mathbb{Z}$. By Lemma 2.1 the set Σ_f is finite and we can assume that elements in Σ_f are ordered as $\{p_1, \cdots, p_m\}$ by its values under f

$$a = f(p_1) < f(p_2) < \cdots < f(p_{m-1}) < f(p_m) = b \text{ [M}_1\text{, section 4]}.$$

Take a $c_i \in (f(p_i), f(p_{i+1}))$, $i \leq m-1$. Then c_i is a regular value of f. As a result $M_i = f^{-1}[a, c_i] \subset M$ is a smooth submanifold with boundary $\partial M_i = L_{c_i}$. Moreover we get a filtration on M by submanifolds

$$M_1 \subset M_2 \subset \cdots \subset M_{m-1} \subset M_m = M.$$

We establish theorem 2 by showing that

1) $M_1 = D^n$;

2) For each k there is an embedding $g : S^{r-1} \to \partial M_k$ so that

$$M_k \cup S(p_{k+1}) = M_k \cup_g D^r,\ r = \text{Ind } (p_{k+1});$$

3) $M_{k+1} = M_k \cup D^r \times D^{n-r}$ with core $M_k \cup_g D^r$.

1) Let $\mathbb{R}^n$ be an Euclidean neighborhood around p_1 so that

$$f(x_1, \cdots, x_n) = a + \Sigma x_i^2,$$

here we have made use of the fact $Ind(p_1) = 0$ (because f attains its absolute minimal value a at p_1) as well as Lemma 2.2. Since $c_1 = a + \varepsilon$ we have

$$f^{-1}[a, c_1] = \{x \in \mathbb{R}^n \mid \| x \|^2 \leq \varepsilon\} \cong D^n.$$

2) With the notation introduced in the proof of Lemma 2.4 we have

$$\text{(C)} \qquad S(p_{k+1}) = \bigcup_{v \in S_-} J_v \cup \{p_{k+1}\}$$

where $S_- \cong S^{r-1}$, $r = Ind(p_{k+1})$, and where J_v is the unique flow line $\varphi_v(t)$ with $\varphi_v(0) = v$ and with $\lim\limits_{t \to +\infty} \varphi_v(t) = p_{k+1}$.

For a $v \in S_-$, $\lim\limits_{t \to -\infty} \varphi_v(t) \in \{p_1, \cdots, p_k\} \subset Int(M_k)$ by (4) and (3) of Lemma 2.3. So J_v must meet ∂M_k at some unique point. The map $g : S_- \to \partial M_k$ such that $g(v) = J_v \cap \partial M_k$ is now well defined and must be an embedding by (2) of Lemma 2.3. We get

$$M_k \cup S(p_{k+1}) = M_k \cup_g D^r$$

form (C).

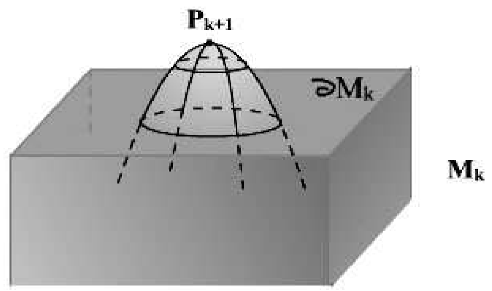

Figure 10 $S_{(P_{k+1})}$ intereseds ∂M_k at an embedded sphere

3). In [M_1, p.33-34], Milnor demonstrated explicitly two deformation retractions

$$r : M_{k+1} \overset{R_1}{\to} M_k \cup D^r \times D^{n-r} \overset{R_2}{\to} M_k \cup S(p_{k+1})$$

where R_1 does not change the diffeomorphism type of M_{k+1} and where $D^r \times D^{n-r}$ is a thickening of the r-cell corresponding to $S(p_{k+1})$.$\square$

3 Morse functions via Euclidean geometry

Our main subject is the effective computation of the additive homology or the multiplicative cohomology of a given manifold M. Recall from section 1 that if M is endowed with a cell decomposition, the homology $H_*(M)$ can be calculated by repeated application of Theorem 1. We have seen further in section 2 that a Morse function f on M may define a cell-decomposition on M with each critical point of index r corresponds to an r-cell in the decomposition. The question that remains to us is

How to find a Morse function on a given manifold?

3–1. Distance function on a Euclidean submanifold

By a classical result of Whitney, every n-dimensional smooth manifold M can be smoothly embedded into Euclidian space of some dimension less than $2n + 1$. Therefore, it suffices to assume that M is a submanifold in an Euclidean space E.

A point $a \in E$ gives rise to a function $f_a : M \to \mathbb{R}$ by $f_a(x) = \| x - a \|^2$.

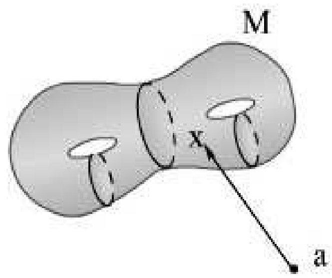

Figure 11 Distance function (from a point) on a Euclidean submanifold M.

Let Σ_a be the set of all critical points of this function. Two questions are:

(a) How to specify the critical set of f_a?

(b) For which choice of the point $a \in E$, f_a is a Morse function on M?

For a point $x \in M$ let $T_xM \subset E$ be the tangent plane to M at x (an affine plane in E with dimension n). Its orthonormal complement

$$\gamma_x = \{v \in E \mid v \perp M_x\}$$

is called *the normal plane to M at x*. We state the answers to questions (a) and (b) in

Lemma 3.1. *Let $f_a : M \to \mathbb{R}$ be as above.*

(1) $\Sigma_a = \{x \in M \mid a - x \in \gamma_x\}$;

(2) For almost all $a \in E$, f_a is a Morse function.

Proof. The function $g_a : E \to \mathbb{R}$ by $x \to \| x - a \|^2$ has gradient field $\text{grad}_x\ g_a = 2(x-a)$. Since $f_a = g_a \mid M$, for a $x \in M$,

$$\text{grad}_x\ f_a = \text{the orthonormal projection of } 2(x-a) \text{ to } T_xM.$$

So $x \in \Sigma_a$ (i.e. $\text{grad}_x\ f_a = 0$) is equivalent to $2(x-a) \perp T_xM$. This shows (1).

Let $\Lambda \subset E$ be the focal set of the submanifold $M \subset E$. It can be shown that f_a is a Morse function if and only if $a \in E \backslash \Lambda$. (2) follows from the fact that Λ has measure 0 in E (cf. [M_2, p.32-38]). □

3–2. Examples of submanifolds in Euclidean spaces

Many manifolds important in geometry are already sitting in Euclidean spaces in some ready-made fashion. We present such examples.

Let $\mathbb{F}$ be one of $\mathbb{R}$ (the field of reals), $\mathbb{C}$ (the field of complex) or $\mathbb{H}$ (the division algebra of quaternions). Let E be one of the following real vector spaces:

the space of $n \times n$ matrices over $\mathbb{F}$: $M(n;\mathbb{F})$;

the space of complex Hermitian matrices:

$$S(n;\mathbb{C}) = \{x \in M(n;\mathbb{C}) \mid x^\tau = x\};$$

the space of complex symmetric matrices

$$S^+(n;\mathbb{C}) = \{x \in M(n;\mathbb{C}) \mid x^\tau = \overline{x}\};$$

the space of real skew symmetric matrices:

$$S^-(2n;\mathbb{R}) = \{x \in M(2n;\mathbb{R}) \mid x^\tau = -x\}.$$

Their dimensions as real vector spaces are respectively

$\dim_{\mathbb{R}} M(n;\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F} \cdot n^2$;

$\dim_{\mathbb{R}} S(n;\mathbb{C}) = n(n+1)$;

$\dim_{\mathbb{R}} S^+(n;\mathbb{C}) = n(n-1)$;

$\dim_{\mathbb{R}} S^-(2n;\mathbb{R}) = n(2n-1)$.

Further, E is an Euclidean space with the metric specified by

$$< x, y >= \mathrm{Re}[Tr(x^* y)],\ x, y \in E,$$

where $*$ means transpose followed by conjugation.

Consider in E the following submanifolds

$O(n;\mathbb{F}) = \{x \in M(n;\mathbb{F}) \mid x^* x = I_n\}$

$G_{n,k} = \{x \in S^+(n;\mathbb{C}) \mid x^2 = I_n,\ l(x) = k\}$;

$LG_n = \{x \in S(n;\mathbb{C}) \mid \overline{x} x = I_n\}$;

$\mathbb{C}S_n = \{x \in S^-(2n;\mathbb{R}) \mid x^2 = -I_{2n}\}$,

where $l(x)$ means "the number of negative eigenvalues of x"and where I_n is the identity matrix. The geometric interests in these manifolds may be illustrated in

$$O(n;\mathbb{F}) = \begin{cases} O(n) \text{ if } \mathbb{F} = \mathbb{R}\text{: the real orthogonal group of rank } n; \\ U(n) \text{ if } \mathbb{F} = \mathbb{C}\text{: the unitary group of rank } n; \\ Sp(n) \text{ if } \mathbb{F} = \mathbb{H}\text{: the symplectic group of rank } n; \end{cases}$$

$G_{n,k}$: the Grassmannian of k-subspaces in $\mathbb{C}^n$;
LG_n: the Grassmannian of Largrangian subspaces in $\mathbb{C}^n$;
$\mathbb{C}S_n$: the Grassmannian of complex structures on $\mathbb{R}^{2n}$.

3–3.Morse functions via Euclidean geometry

Let $0 < \lambda_1 < \cdots < \lambda_n$ be a sequence of n reals, and let $a \in E$ be the point with

$$a = \left\{ \begin{array}{l} diag\{\lambda_1, \cdots, \lambda_n\} \text{ if } M \neq \mathbb{C}S_n; \\ \lambda_1 J \oplus \cdots \oplus \lambda_n J,\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ if } M = \mathbb{C}S_n. \end{array} \right.$$

With respect to the metric on E specified in 3-2, the function

$$f_a : M \to \mathbb{R},\ f_a(x) = \| x - a \|^2$$

admits a simple-looking expression

$$f_a((x_{ij})) = < x, x > + < a, a > -2 < a, x >$$

$$= const - 2\{ \begin{array}{l} \Sigma\lambda_i \operatorname{Re}(x_{ii}) \text{ if } M = G_{n,k},\ O(n;\mathbb{F}),\ LG_n; \text{ and} \\ \Sigma\lambda_i x_{2i-1,2i} \text{ if } M = \mathbb{C}S_n. \end{array}$$

For a subsequence $I = [i_1, \cdots, i_r] \subseteq [1, \cdots, n]$, denote by $\sigma_I \in E$ the point

$$\sigma_I = \{ \begin{array}{l} diag\{\varepsilon_1, \cdots, \varepsilon_n\} \text{ if } M \neq \mathbb{C}S_n; \\ \varepsilon_1 J \oplus \cdots \oplus \varepsilon_n J \text{ if } M = \mathbb{C}S_n, \end{array}$$

where $\varepsilon_k = -1$ if $k \in I$ and $\varepsilon_k = 1$ otherwise.

Theorem 3. *In each of the above four cases, $f_a : M \to R$ is a Morse function on M. Further,*
(1) the set of critical points of f_a is

$$\Sigma_a = \{ \begin{array}{l} \{\sigma_0,\ \sigma_I \in M \mid I \subseteq [1, \cdots, n]\} \text{ if } M \neq G_{n,k}; \\ \{\sigma_I \in M \mid I \subseteq [1, \cdots, n] \text{ with } \mid I \mid = k\} \text{ if } M = G_{n,k}. \end{array}$$

(2) the index functions are given respectively by

$$Ind(\sigma_{i_1,\cdots,i_r}) = \{ \begin{array}{l} \dim_{\mathbb{R}} \mathbb{F} \cdot (i_1 + \cdots + i_r) - r \text{ if } M = O(n;\mathbb{F}); \\ 2(i_1 + \cdots + i_r - r) \text{ if } M = \mathbb{C}S_n; \\ i_1 + \cdots + i_r \text{ if } M = LG_n; \end{array}$$

$$Ind(\sigma_{i_1,\cdots,i_k}) = 2 \sum_{1 \leq s \leq k} (i_s - s) \text{ if } M = G_{n,k}.$$

3–4. Proof of Theorem 3

We conclude Section 3 by a proof of Theorem 3.

Lemma 3.2. *For a $x \in M$ one has*

$$T_xM = \left\{ \begin{array}{l} \{u \in E \mid xu = -ux\} \text{ for } M = G_{n,k};\ \mathbb{C}S_n \\ \{u \in E \mid x^*u = -u^*x\} \text{ for } M = O(n;\mathbb{F}) \\ \{u \in E \mid \overline{x}u = -\overline{u}x\} \text{ for } M = LG_n. \end{array} \right.$$

Consequently

$$\gamma_xM = \left\{ \begin{array}{l} \{u \in E \mid xu = ux\} \text{ for } M = G_{n,k};\ \mathbb{C}S_n \\ \{u \in E \mid x^*u = u^*x\} \text{ for } M = O(n;\mathbb{F}) \\ \{u \in E \mid \overline{x}u = \overline{u}x\} \text{ for } M = LG_n. \end{array} \right.$$

Proof. We verify Lemma 3.2 for the case $M = G_{n,k}$ as an example. Consider the map $h : S^+(n;\mathbb{C}) \to S^+(n;\mathbb{C})$ by $x \to x^2$. Then

(1) $h^{-1}(I_n) = \bigsqcup_{1 \le t \le n-1} G_{n,t}$;

(2) the tangent map of h at a point $x \in S^+(n;\mathbb{C})$ is

$$T_xh(u) = \lim_{t \to 0} \tfrac{h(x+tu)-h(x)}{t} = ux + xu.$$

It follows that, for a $x \in G_{n,k}$,

$$T_xG_{n,k} \subseteq KerT_xh = \{u \in S^+(n;\mathbb{C}) \mid ux + xu = 0\}.$$

On the other hand $\dim_{\mathbb{C}} KerT_xh = k(n-k)$ $(= \dim_{\mathbb{C}} T_xG_{n,k})$. So the dimension comparison yields

$$T_xG_{n,k} = \{u \in S^+(n;\mathbb{C}) \mid xu = -ux\}.$$

For any $x \in G_{n,k}$ the ambient space $E = S^+(n;\mathbb{C})$ admits the orthogonal decomposition

$$S^+(n;\mathbb{C}) = \{u \mid xu = -ux\} \oplus \{u \mid xu = ux\}$$

in which the first summand has been identified with $T_xG_{n,k}$ in the above computation. It follows that $\gamma_xG_{n,k} = \{u \mid xu = ux\}$.

The other cases can be verified by the same method.□

Lemma 3.3. *Statement (1) of Theorem 3 holds true.*

Proof. Consider the case $G_{n,k} \subset S^+(n;\mathbb{C})$.

$x \in \Sigma_a \Leftrightarrow x - a \in \gamma_x G_{n,k}$ (by (1) of Lemma 3.1)
$\Leftrightarrow (x-a)x = x(x-a)$ (by Lemma 3.2)
$\Leftrightarrow xa = ax.$

Since a is diagonal with the distinguished diagonal entries $\lambda_1 < \cdots < \lambda_n$, x is also diagonal. Since $x^2 = I_n$ with $l(x) = k$, we must have $x = \sigma_{i_1,\cdots,i_k}$ for some $[i_1, \cdots, i_k] \subseteq [1, \cdots, n]$.

Analogous computations verify the other cases.□

To prove Theorem 3 we need examining the Hessian operator $H_{x_0}(f_a) : T_{x_0}M \to T_{x_0}M$ at a critical point $x_0 \in \Sigma_a$. The following formulae will be useful for this purpose.

Lemma 3.4. $H_{x_0}(f_a)(u) = \begin{cases} (ua - au)x_0 & \text{for } M = G_{n,k}; \mathbb{C}S_n; \\ (u^*a - au^*)x_0 & \text{for } M = O(n;\mathbb{F}); \\ (\overline{u}a - a\overline{u})x_0 & \text{for } M = LG_n. \end{cases}$

Proof. As a function on the Euclidean space E, f_a has gradient field $2(x-a)$. However, the gradient field of the restricted function $f_a \mid M$ is the orthogonal projection of $2(x-a)$ to T_xM.

In general, for any $x \in M$, a vector $u \in E$ has the "canonical" decomposition

$$u = \begin{cases} \frac{u-xux}{2} + \frac{u+xux}{2} & \text{if } M = G_{n,k}; \mathbb{C}S_n; \\ \frac{u-x^*ux}{2} + \frac{u+x^*ux}{2} & \text{if } M = O(n;\mathbb{F}); \\ \frac{u-\overline{x}ux}{2} + \frac{u+\overline{x}ux}{2} & \text{if } M = LG_n. \end{cases}$$

with the first component in the T_xM and the second component in $\gamma_x M$ by Lemma 3.2. Applying these to $u = 2(x-a)$ yields respectively that

$$grad_x f_a = \begin{cases} (xax - a) & \text{for } M = G_{n,k}; \mathbb{C}S_n; \\ (x^*ax - a) & \text{for } M = O(n;\mathbb{F}); \\ (\overline{x}ax - a) & \text{for } M = LG_n. \end{cases}$$

Finally, the Hessian operator can be computed in term of the gradient as

$$H_{x_0}(f_a)(u) = \lim_{t\to 0} \frac{grad_{x_0+tu} f_a - grad_{x_0} f_a}{t},\ u \in T_xM.$$

As an example we consider the case $M = G_{n,k}$. We have

$$\lim_{t\to 0} \frac{grad_{x_0+tu} f_a - grad_{x_0} f_a}{t} = \lim_{t\to 0} \frac{[(x_0+tu)a(x_0+tu)-a]-[x_0ax_0-a]}{t}$$
$= uax_0 + x_0au = uax_0 + ax_0u$ (because a and x_0 are diagonal)
$= (ua - au)x_0$

(because vectors in $T_{x_0}G_{n,k}$ anti-commute with x_0 by Lemma 3.2).□

Proof of Theorem 3. In view of Lemma 3.3, Theorem 3 will be completed once we have shown

(a) f_a is non-degenerate at any $x_0 \in \Sigma_a$; and

(b) the index functions on Σ_a is given as that in (2) of Theorem 3.

This can be done by applying Lemma 3.2 and Lemma 3.4. We verify these for the cases $M = G_{n,k}$, $O(n)$ and LG_n in detail, and leave the other cases to the reader.

Case 1. $M = G_{n,k} \subset S^+(n;\mathbb{C})$.

(1) The most convenient vectors that span the real vector space $S^+(n;\mathbb{C})$ are

$$\{b_{s,t} \mid 1 \leq s,t \leq n\} \sqcup \{c_{s,t} \mid 1 \leq s \neq t \leq n\},$$

where $b_{s,t}$ has the entry 1 at the places (s,t), (t,s) and 0 otherwise, and where $c_{s,t}$ has the pure imaginary i at (s,t), $-i$ at the (t,s) and 0 otherwise.

(2) For a $x_0 = \sigma_I \in \Sigma_a$, those $b_{s,t}$, $c_{s,t}$ that "*anti-commute*" with x_0 belong to $T_{x_0}G_{n,k}$ by Lemma 3.2, and form a basis for $T_{x_0}G_{n,k}$

$$T_{x_0}G_{n,k} = \{b_{s,t}, c_{s,t} \mid (s,t) \in I \times J\},$$

where J is the complement of I in $[1,\cdots,n]$.

(3) Applying the Hessian (Lemma 3.4) to the $b_{s,t}, c_{s,t} \in T_{x_0}G_{n,k}$ yields

$$H_{x_0}(f_a)(b_{s,t}) = (\lambda_t - \lambda_s)b_{s,t};$$
$$H_{x_0}(f_a)(c_{s,t}) = (\lambda_t - \lambda_s)c_{s,t}.$$

That is, the $b_{s,t}, c_{s,t} \in T_{x_0}G_{n,k}$ are precisely the eigenvectors for the operator $H_{x_0}(f_a)$. These indicate that $H_{x_0}(f_a)$ is nondegenerate (since $\lambda_t \neq \lambda_s$ for all $s \neq t$), hence f_a is a Morse function.

(4) It follows from the formulas in (3) that the negative space for $H_{x_0}(f_a)$ is spanned by $\{b_{s,t},\ c_{s,t} \mid (s,t) \in I \times J, t < s\}$. Consequently

$$Ind(\sigma_I) = 2\#\{(s,t) \in I \times J \mid t < s\} = 2\sum_{1\leq s\leq k}(i_s - s).$$

Case 2. $M = O(n) \subset M(n;\mathbb{R})$.

(1) A natural set of vectors that spans the space $M(n;\mathbb{R})$ is

$$\{b_{s,t} \mid 1 \le s \le t \le n\} \sqcup \{\beta_{s,t} \mid 1 \le s < t \le n\},$$

where $b_{s,t}$ is as case 1, and where $\beta_{s,t}$ is the skew symmetric matrix with entry 1 at the (s,t) place, -1 at the (t,s) place and 0 otherwise;

(2) For a $x_0 = \sigma_I \in \Sigma_a$ those $b_{s,t}$, $\beta_{s,t}$ that"anti-commute" with x_0 yields precisely a basis for

$$T_{x_0}O(n) = \{\beta_{s,t} \mid (s,t) \in I \times I, J \times J, s < t\} \sqcup \{b_{s,t} \mid (s,t) \in I \times J\}$$

by Lemma 3.2, where J is the complement of I in $[1,\cdots,n]$.

(3) Applying the Hessian operator (Lemma 3.4) to $b_{s,t}$, $\beta_{s,t} \in T_{x_0}O(n)$ tells

$$H_{x_0}(f_a)(\beta_{s,t}) = \{ \begin{array}{l} -(\lambda_t + \lambda_s)\beta_{s,t} \text{ if } (s,t) \in I \times I, s < t; \\ (\lambda_t + \lambda_s)\beta_{s,t} \text{ if } (s,t) \in J \times J,\ s < t. \end{array}$$
$$H_{x_0}(f_a)(b_{s,t}) = (\lambda_t - \lambda_s)b_{s,t} \text{ if } (s,t) \in I \times J.$$

This implies that the $b_{s,t}$, $\beta_{s,t} \in T_{x_0}G_{n,k}$ are precisely the eigenvectors for the operator $H_{x_0}(f_a)$, and the f_a is a Morse function.

(4) It follows from the computation in (3) that

$Ind(\sigma_I) = \#\{(s,t) \in I \times I \mid s < t\} + \#\{(s,t) \in I \times J \mid t < s\}$

$$= 1 + 2 + \cdots + (r-1) + [(i_1 - 1) + (i_2 - 2) + \cdots + (i_r - r)]$$
$$= \Sigma i_s - r.$$

Case 3. $M = LG_n \subset S(n;\mathbb{C})$.

(1) Over reals, the most natural vectors that span the space $S(n;C)$ are

$$\{b_{s,t} \mid 1 \le s,t \le n\} \cup \{ib_{s,t} \mid 1 \le s,t \le n\},$$

where $b_{s,t}$ is as that in Case 1 and where i is the pure imaginary;

(2) For a $x_0 = \sigma_I \in \Sigma_a$ those "anti-commute" with x_0 yields precisely a basis for $T_{x_0}LG_n$

$$T_{x_0}LG_n = \{b_{s,t} \mid (s,t) \in I \times J \amalg J \times I\} \sqcup$$
$$\{ib_{s,t} \mid (s,t) \in I \times I \sqcup J \times J\}$$

where J is the complement of I in $[1, \cdots, n]$.

(3) Applying the Hessian to $b_{s,t}$, $ib_{s,t} \in T_{x_0}LG_n$ (cf. Lemma 3.4) tells

$$H_{x_0}(f_a)(ib_{s,t}) = \{ \begin{array}{c} -(\lambda_t + \lambda_s)ib_{s,t} \text{ if } (s,t) \in I \times I \\ (\lambda_t + \lambda_s)ib_{s,t} \text{ if } (s,t) \in J \times J \end{array} ;$$
$$H_{x_0}(f_a)(b_{s,t}) = \{ \begin{array}{c} (\lambda_t - \lambda_s)b_{s,t} \text{ if } (s,t) \in I \times J \\ (\lambda_s - \lambda_t)b_{s,t} \text{ if } (s,t) \in J \times I \end{array} .$$

It follows that the $b_{s,t}$, $ib_{s,t} \in T_{x_0}G_{n,k}$ are precisely the eigenvectors for the operator $H_{x_0}(f_a)$, and f_a is a Morse function.

(4) It follows from (2) and (3) that

$$Ind(\sigma_I) = \#\{(s,t) \in I \times I \mid t \leq s\} + \#\{(s,t) \in I \times J \mid t \leq s\}$$

$$= i_1 + \cdots + i_r. \square$$

Remark. Let E be one of the following matrix spaces:
the space of $n \times k$ matrices over $\mathbb{F}$: $M(n \times k; \mathbb{F})$;
the space of symmetric matrices $S^+(n; \mathbb{F}) = \{x \in M(n; \mathbb{F}) \mid x^\tau = \overline{x}\}$.
Consider in E the following submanifolds:

$$V_{n,k}(\mathbb{F}) = \{x \in M(n \times k; \mathbb{F}) \mid \overline{x}^\tau x = I_k\};$$
$$G_{n,k}(\mathbb{F}) = \{x \in S^+(n; \mathbb{F}) \mid x^2 = I_n,\, l(x) = k\}.$$

These are known respectively as the *Stiefel manifold* of orthonormal k-frames on $\mathbb{F}^n$ (the n-dimensional $\mathbb{F}$-vector space) and the *Grassmannian* of k-dimensional $\mathbb{F}$-subspaces in $\mathbb{F}^n$. Results analogous to Theorem 3 hold for these two family of manifolds as well [D_1], [D_2].

Remark. In [VD, Theorem 1.2], the authors proved that the function f_a on $M = G_{n,k}(\mathbb{F})$, LG_n, $\mathbb{C}S_n$ is a perfect Morse function (without specifying the set Σ_a as well as the index function Ind: $\Sigma_a \to \mathbb{Z}$).

4 Morse functions of Bott-Samelson type

We recall the original construction of Bott-Samelson cycles in 4–1 and explain its generalization due to Hsiang-Palais-Terng [HTP] in 4–2.

In fact, the Morse functions concerned in Theorem 3 are all Bott-Samelson type (cf. Theorem 6). The induced cohomology homomorphism of Bott-Samelson cycles enables one to resolve the multiplication in cohomology into the multiplication of symmetric functions of various types (Theorem 7).

4–1. Morse functions on flag manifolds (cf. [BS_1,BS_2]).

Let G be a compact connected semi-simple Lie group with the unit $e \in G$ and a fixed maximal torus $T \subset G$. The tangent space $L(G) = T_eG$ (resp. $L(T) = T_eT$) is canonically furnished with the structure of an algebra, known as the *Lie algebra* (resp. the *Cartan subalgebra*) of G. The exponential map induces the commutative diagram

$$\begin{array}{ccc} L(T) & \to & L(G) \\ \exp\downarrow & & \downarrow\exp \\ T & \to & G \end{array}$$

where the horizontal maps are the obvious inclusions. Equip $L(G)$ (hence also $L(T)$) an inner product invariant under the adjoint action of G on $L(G)$.

For a $v \in L(T)$ let $C(v)$ be the centralizer of $\exp(v) \in G$. The set of singular points in $L(T)$ is the subspace of the Cartan subalgebra $L(T)$:

$$\Gamma = \{v \in L(T) \mid \dim C(v) > \dim T\}.$$

Lemma 4.1. *Let* $m = \frac{1}{2}(\dim G - \dim T)$. *There are precisely* m *hyperplanes* $L_1, \cdots, L_m \subset L(T)$ *through the origin* $0 \in L(T)$ *so that* $\Gamma = \underset{1\le i\le m}{\cup} L_i$.$\square$

The planes $L_1, \cdots, L_m$ are known as the singular planes of G. It divide $L(T)$ into finite many convex hulls, known as the *Weyl chambers* of G. Reflections in these planes generate the *Weyl group* W of G.

Fix a regular point $a \in L(T)$. The adjoint representation of G gives rise to a map $G \to L(G)$ by $g \to Ad_g(a)$, which induces an embedding of the *flag manifold* $G/T = \{gT \mid g \in G\}$ of left cosets of T in G into $L(G)$. In this way G/T becomes a submanifold in the Euclidean space $L(G)$.

Consider the function $f_a : G/T \to \mathbb{R}$ by $f_a(x) = \parallel x - a \parallel^2$. The following beautiful result of Bott and Samelson [BS_1,BS_2] tells how to read the critical points information of f_a from the linear geometry of the vector space $L(T)$.

Theorem 4. *f_a is a Morse function on G/T with critical set*

$$\Sigma_a = \{w(a) \in L(T) \mid w \in W\}$$

(the orbit of the W-action on $L(T)$ through the point $a \in L(T)$).

The index function Ind: $\Sigma_a \to \mathbb{Z}$ is given by

$$Ind(w(a)) = 2\#\{L_i \mid L_i \cap [a, w(a)] \neq \emptyset],$$

where $[a, w(a)]$ is the segment in $L(T)$ from a to $w(a)$.

Moreover, Bott and Samelson constructed a set of geometric cycles in G/T that realizes an additive basis of $H_*(G/T;\mathbb{Z})$ as follows.

For a singular plane $L_i \subset L(T)$ let $K_i \subset G$ be the centralizer of $\exp(L_i)$. The Lie subgroup K_i is *very simple* in the sense that $T \subset K_i$ is also a maximal torus with the quotient K_i/T diffeomorphic to the 2-sphere S^2.

For a $w \in W$ assume that the singular planes that meet the directed segments $[a, w(a)]$ are in the order $L_1, \cdots, L_r$. Put $\Gamma_w = K_1 \times_T \cdots \times_T K_r$, where the action of $T \times \cdots \times T$ (r-copies) acts on $K_1 \times \cdots \times K_r$ from the left by

$$(k_1, \cdots, k_r)(t_1, \cdots, t_r) = (k_1 t_1, t_1^{-1} k_2 t_2, \cdots, t_{r-1}^{-1} k_r t_r).$$

The map $K_1 \times \cdots \times K_r \to G/T$ by

$$(k_1, \cdots, k_r) \to Ad_{k_1 \cdots k_r}(w(a))$$

clearly factors through the quotient manifold Γ_w, hence induces a map

$$g_w : \Gamma_w \to G/T.$$

Theorem 5. *The homology $H_*(G/T;\mathbb{Z})$ is torsion free with the additive basis $\{g_{w*}[\Gamma_w] \in H_*(G/T;\mathbb{Z}) \mid w \in W\}$.*

Proof. Let $e \in K_i(\subset G)$ be the group unit and put $\overline{e} = [e, \cdots, e] \in \Gamma_w$. It were actually shown by Bott and Samelson that

(1) $g_w^{-1}(w(a))$ consists of the single point $\overline{e}$;

(2) the composed function $f_a \circ g_w : \Gamma_w \to \mathbb{R}$ attains its maximum only at $\overline{e}$;

(3) the tangent map of g_w at $\overline{e}$ maps the tangent space of Γ_w at $\overline{e}$ isomorphically onto the negative part of $H_{w(a)}(f_a)$.

The proof is completed by Lemma 4.2 in **4.2**.□

Remark. It was shown by Chevalley in 1958 [Ch] that the flag manifold G/T admits a cell decomposition $G/T = \underset{w\in W}{\cup} X_w$ indexed by elements in W, with each cell X_w an algebraic variety, known as a Schubert variety on G/T. Hansen [Han] proved in 1971 that $g_w(\Gamma_w) = X_w$, $w \in W$. So the map g_w is currently known as the "*Bott-Samelson resolution* of X_w".

For the description of Bott-Samelson cycles and their applications in Algebro-geometric setting, see M. Brion [Br] in this volume.

4–2. Morse function of Bott-Samelson type

In differential geometry, the study of *isoparametric submanifolds* was begun by E. Cartan in 1933. In order to generalize Bott-Samelson's above cited results to these manifolds Hsiang, Palais and Terng introduced the following notation in their work [HPT][1].

Definition 4.1. *A Morse function $f : M \to R$ on a smooth closed manifold is said to be of Bott-Samelson type over $\mathbb{Z}_2$(resp. $\mathbb{Z}$) if for each $p \in \Sigma_f$ there is a map (called a Bott-Samelson cycle of f at p)*

$$g_p : N_p \to M$$

where N_p is a closed oriented (resp. unoriented) manifold of dimension $Ind(p)$ and where

(1) $g_p^{-1}(p) = \{\overline{p}\}$ (a single point);

(2) $f \circ g_p$ attains absolute maximum only at $\overline{p}$;

(3) the tangent map $T_{\overline{p}} g_p : T_{\overline{p}} N_p \to T_p M$ is an isomorphism onto the negative space of $H_p(f)$.

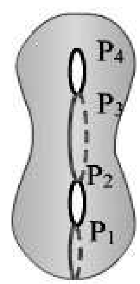

Figure 12 Bott — Samelson cycles of dimension 1 in double torus.

Information that one can get from a Morse function of Bott-Samelson type can be seen from the next result [HPT].

Lemma 4.2. *If $f : M \to R$ is a Morse function of Bott-Samelson type with Bott-Samelson cycles $\{g_p : N_p \to M \mid p \in \Sigma_f\}$, then $H_*(M;\mathbb{Z})$ (resp. $H_*(M;\mathbb{Z}_2)$) has the additive basis*

$$\{g_{p*}[N_p] \in H_*(M;\mathbb{Z}) \mid p \in \Sigma_f\}$$
$$(\text{resp. } \{g_{p*}[N_p]_2 \in H_*(M;\mathbb{Z}_2) \mid p \in \Sigma_f\}),$$

[1] In fact, the embedding $G/T \subset L(G)$ described in 4-1 defines G/T as an isoparametric submanifold in $L(G)$ [HPT].

where $g_{p*} : H_*(N_p;\mathbb{Z}) \to H_*(M;\mathbb{Z})$ is the induced homomorphism and where $[N_p] \in H_*(N_p;\mathbb{Z})$ (resp. $[N_p]_2 \in H_*(N_p;\mathbb{Z}_2)$) is the orientation class (resp. $\mathbb{Z}_2$-orientation class).

Proof. Without loss of generalities we may assume (as in the proof of Theorem 2) that $\Sigma_f = \{p_1, \cdots, p_m\}$ and that $f(p_k) < f(p_{k+1})$, $1 \le k \in m-1$. Consider the filtration on M: $M_1 \subset M_2 \subset \cdots \subset M_m = M$ defined by f and Σ_f such that $M_{k+1} \backslash M_k$ contains p_k for every $1 \le k \le m-1$.

It suffices to show, that if we put $p = p_{k+1}$, $m = Ind(p)$, then

$$\text{(D)} \qquad H_r(M_{k+1};\mathbb{Z}) = \begin{cases} H_r(M_k;\mathbb{Z}) \text{ if } r \neq m; \\ H_r(M_k;\mathbb{Z}) \oplus \mathbb{Z} \text{ if } r = m, \end{cases}$$

where the summand $\mathbb{Z}$ is generated by $g_{p*}[N_p]$.

The Bott-Samelson cycle $g_p : N_p \to M$ (cf. Definition 4.1) is clearly a map into M_{k+1}. Let $r : M_{k+1} \to M_k \cup D^m$ be the strong deformation retraction from the proof of Theorem 2, and consider the composed map

$$g : N_p \overset{g_p}{\to} M_{k+1} \overset{r}{\to} M_k \cup D^m.$$

The geometric constraints (1)-(3) on the Bott-Samelson cycle g_p imply that there exists an Euclidean neighborhood $U \subset D^m$ centered at $p = 0 \in D^m$ so that if $V =: g^{-1}(U)$, then g restricts to a diffeomorphism $g \mid V : V \to U$. The proof of (D) (hence of Lemma 4.2) is clearly done by the exact ladder induced by the "*relative homeomorphism*" $g : (N_p, N_p\backslash V) \to (M_k \cup D^m, M_k \cup D^m \backslash U)$

$$\begin{array}{ccccccccc} & & & & \mathbb{Z} & & \mathbb{Z} & & \\ & & & & \| & & \| & & \\ & & 0 \to H_m(N_p) & \overset{\cong}{\to} & H_m(N_p, N_p\backslash V) & \to & H_{d-1}(N_p\backslash V) & \to \cdots & \\ & & & & g_* \downarrow & & g_* \downarrow\cong & & \\ 0 \to H_d(M_k) \to & & H_d((M_k \cup D^m) & \to & H_d((M_k \cup D^m, M_k) & \to & H_{d-1}(M_k) & \to \cdots & \end{array}$$

.□

4–3. Bott-Samelson cycles and resolution of Schubert varieties

Le M be one of the following manifolds

$O(n;\mathbb{F})$: orthogonal (or unitary, or symplectic) group of rank n;

$\mathbb{C}S_n$: the Grassmannian of complex structures on $\mathbb{R}^{2n}$;

$G_{n,k}$: the Grassmannian of k-linear subspaces on $\mathbb{C}^n$

and

LG_n: the Grassmannian of Lagrangian subspaces on $\mathbb{C}^n$.

Let $f_a : M \to \mathbb{R}$ be the Morse function considered in Theorem 3 of §3.

Theorem 6. *In each case f_a is a Morse function of Bott-Samelson type which is*

(1) over $\mathbb{Z}$ for $M = U(n), Sp(n), \mathbb{C}S_n, G_{n,k}$;

(2) over $\mathbb{Z}_2$ for $M = O(n)$ and LG_n.

Instead of giving a proof of this result I would like to show the geometric construction of the Bott-Samelson cycles required to justify the theorem, and to point out the consequences which follow up (cf. Theorem 7).

Let $\mathbb{R}P^{n-1}$ be the real projective space of lines through the origin 0 in $\mathbb{R}^n$; $\mathbb{C}P^{n-1}$ the complex projective space of complex lines through the origin 0 in $\mathbb{C}^n$, and let $G_2(\mathbb{R}^{2n})$ be the Grassmannian of oriented 2-planes through the origin in $\mathbb{R}^{2n}$.

Construction 1. Resolution $h : \widetilde{M} \to M$ of M.

(1) If $M = SO(n)$ (the special orthogonal group of order n) we let $\widetilde{M} = \mathbb{R}P^{n-1} \times \cdots \times \mathbb{R}P^{n-1}$ (n'-copies, where $n' = 2[\frac{n}{2}]$) and define the map $h : \widetilde{M} \to M$ to be

$$h(l_1, \cdots, l_{n'}) = \Pi_{1 \leq i \leq n'} R(l_i),$$

where $l_i \in \mathbb{R}P^{n-1}$ and where $R(l_i)$ is the reflection on $\mathbb{R}^n$ in the hyperplane $l_i^{\perp}$ orthogonal to l_i.

(2) If $M = G_{n,k}$ we let

$$\widetilde{M} = \{(l_1, \cdots, l_k) \in \mathbb{C}P^{n-1} \times \cdots \times \mathbb{C}P^{n-1} \mid l_i \perp l_j\} \text{ (k-copies)}$$

and define the map $h : \widetilde{M} \to M$ to be $h(l_1, \cdots, l_k) =< l_1, \cdots, l_k >$, where $l_i \in \mathbb{C}P^{n-1}$ and where $< l_1, \cdots, l_k >$ means the k-plane spanned by the $l_1, \cdots, l_k$.

(3) If $M = \mathbb{C}S_n$ we let

$$\widetilde{M} = \{(L_1, \cdots, L_n) \in G_2(\mathbb{R}^{2n}) \times \cdots \times G_2(\mathbb{R}^{2n}) \mid L_i \perp L_j\} \text{ (n-copies)}$$

and define the map $h : \widetilde{M} \to M$ to be $h(L_1, \cdots, L_k) = \Pi_{1 \leq i \leq n} \tau(L_i)$, where $L_i \in G_2(\mathbb{R}^{2n})$ and where $\tau(L_i) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the isometry which fixes points in the orthogonal complements $L_i^{\perp}$ of L_i and is the $\frac{\pi}{2}$ rotation on L_i in accordance with the orientation.

Construction 2. Bott-Samelson cycles for the Morse function $f_a : M \to \mathbb{R}$ (cf. [section3, Theorem 3]).

(1) If $M = SO(n)$ then $\Sigma_a = \{\sigma_0, \, \sigma_I \in M \mid I \subseteq [1, \cdots, n], \mid I \mid \leq n'\}$. For each $I = (i_1, \cdots, i_r) \subseteq [1, \cdots, n]$ we put

$$\mathbb{R}P[I] = \mathbb{R}P^0 \times \cdots \times \mathbb{R}P^0 \times \mathbb{R}P^{i_1} \times \cdots \times \mathbb{R}P^{i_r} \ (n'\text{-copies}).$$

Since $\mathbb{R}P[I] \subset \widetilde{M}$ we may set $h_I = h \mid \mathbb{R}P[I]$.

The map $h_I : \mathbb{R}P[I] \to SO(n)$ is a Bott-Samelson cycle for f_a at σ_I.

(2) If $M = G_{n,k}$ then $\Sigma_a = \{\sigma_I \in M \mid I = (i_1, \cdots, i_k) \subseteq [1, \cdots, n]\}$. For each $I = (i_1, \cdots, i_k) \subseteq [1, \cdots, n]$ we have

$$\mathbb{C}P^{i_1} \times \cdots \times \mathbb{C}P^{i_k}, \widetilde{M} \subset \mathbb{C}P^{n-1} \times \cdots \times \mathbb{C}P^{n-1} (k\text{-copies}).$$

So we may define the intersection $\mathbb{C}P[I] = \mathbb{C}P^{i_1} \times \cdots \times \mathbb{C}P^{i_k} \cap \widetilde{M}$ in $\mathbb{C}P^{n-1} \times \cdots \times \mathbb{C}P^{n-1}$ and set $h_I = h \mid \mathbb{C}P[I]$.

The map $h_I : \mathbb{C}P[I] \to G_{n,k}$ is a Bott-Samelson cycle for f_a at σ_I.

4–4. Multiplication in cohomology: *Geometry versus combinatorics*

Up to now we have plenty examples of Morse functions of Bott-Samelson type. Let $f : M \to \mathbb{R}$ be such a function with critical set $\Sigma_f = \{p_1, \cdots, p_m\}$. From the proof of Lemma 4.2 we see that each descending cell $S(p_i) \subset M$ forms a closed cycle on M and all of them form an additive basis for the homology

$$\{[S(p_i)] \in H_{r_i}(M; \mathbb{Z} \text{ or } \mathbb{Z}_2) | \ 1 \leq i \leq m, \, r_i = Ind(p_i)\},$$

where the coefficients in homology depend on whether the Bott-Samelson cycles are orientable or not.

Many pervious work on Morse functions stopped at this stage, for people were content to have found Morse functions on manifolds whose critical points determine an additive basis for homology (such functions are normally called *perfect Morse functions*).

However, the difficult task that one has experienced in topology is not to find an additive basis for homology, but is to understand the multiplicative rule among basis elements in cohomology. More precisely, we let

$$\{[\Omega(p_i)] \in H^{r_i}(M; \mathbb{Z} \text{ or } \mathbb{Z}_2) \mid 1 \leq i \leq m, \, r_i = Ind(p_i)\}$$

be the basis for the cohomology Kronecker dual to the $[S(p_i)]$ as

$$< [\Omega(p_i)], [S(p_j)] >= \delta_{ij}.$$

Then we must have the expression

$$[\Omega(p_i)] \cdot [\Omega(p_j)] = \Sigma a_{ij}^k [\Omega(p_k)]$$

in the ring $H^*(M; \mathbb{Z}$ or $\mathbb{Z}_2)$, where $a_{ij}^k \in \mathbb{Z}$ or $\mathbb{Z}_2$ depending on whether the Bott-Samelson cycles orientable or not, and where $\cdot$ means *intersection product* in Algebraic Geometry and *cup product* in Topology.

Problem 4. *Find the numbers* a_{ij}^k *for each triple* $1 \leq i, j, k \leq m$.

To emphasis Problem 4 we quote from N. Steenrod [St, p.98]:

> "the cup product requires a diagonal approximation $d_\# : M \to M \times M$. Many difficulties experienced with the cup product in the past arose from the great variety of choices of $d_\#$, any particular choice giving rise to artificial looking formulas".

We advise alos the reader to consult [La], [K], and [S] for details on multiplicative rules in the intersection ring of $G_{n,k}$ in algebraic geometry, and their history.

Bott-Samelson cycles provide a way to study Problem 4. To explain this we turn back to the constructions in 4-3. We observe that

(i) The resolution $\widetilde{M}$ of M are constructed from the most familiar manifolds as

$\mathbb{R}P^{n-1}$ =the real projective space of lines through the origin in $\mathbb{R}^n$;

$\mathbb{C}P^{n-1}$ =the real projective space of lines through the origin in $\mathbb{C}^n$;

$G_2(\mathbb{R}^{2n})$ =the Grassmannian of oriented 2-dimensional subspaces in $\mathbb{R}^{2n}$ and whose cohomology are well known as

$$H^*(\mathbb{R}P^{n-1}; \mathbb{Z}_2) = \mathbb{Z}_2[t]/t^n; \qquad H^*(\mathbb{C}P^{n-1}; \mathbb{Z}) = \mathbb{Z}[x]/x^n;$$

$$H^*(G_2(\mathbb{R}^{2n}); \mathbb{Z}) = \begin{cases} \mathbb{Z}[y, v]/ < x^n - 2x \cdot v, v^2 > & \text{if } n \equiv 1 \bmod 2; \\ \mathbb{Z}[y, v]/ < x^n - 2x \cdot v, v^2 - x^{n-1} \cdot v > & \text{if } n \equiv 0 \bmod 2 \end{cases}$$

where

(a) $t(\in H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2))$ is the Euler class for the canonical real line bundle over $\mathbb{R}P^{n-1}$;

(b) $x(\in H^2(\mathbb{C}P^{n-1}; \mathbb{Z}))$ is the Euler class of the real reduction for the canonical complex line bundle over $\mathbb{C}P^{n-1}$;

(c) $y(\in H^2(G_2(\mathbb{R}^{2n});\mathbb{Z}))$ is the Euler class of the canonical oriented real 2-bundle γ over $G_2(\mathbb{R}^{2n})$, and where if $s \in H^{2n-2}(G_2(\mathbb{R}^{2n});\mathbb{Z})$ is the Euler class for the orthogonal complement ν of γ in $G_2(\mathbb{R}^{2n})\times \mathbb{R}^{2n}$, then

$$v = \tfrac{1}{2}(y^{n-1} + s) \in H^{2n-2}(G_2(\mathbb{R}^{2n});\mathbb{Z})^2.$$

(ii) the manifolds $\widetilde{M}$ are simpler than M either in terms of their geometric formation or of their cohomology

$H^*(\widetilde{M};\mathbb{Z}) = \mathbb{Z}_2[t_1,\cdots,t_{n'}]/ < t_i^n, 1 \le i \le n' >$ if $M = SO(n)$;

$H^*(\widetilde{M};\mathbb{Z}) = \mathbb{Z}[x_1,\cdots,x_k]/ < p_i, 1 \le i \le k >$ if $M = G_{n,k}$; and

$H^*(\widetilde{M};\mathbb{Z}) = \mathbb{Q}[y_1,\cdots,y_n]/ < e_i(y_1^2,\cdots,y_n^2), 1 \le i \le n-1; y_1\cdots y_n) >$ if $\widetilde{M} = \mathbb{C}S_n$, where p_i is the component of the formal polynomial

$$\prod_{1\le s\le i}(1+x_s)^{-1}$$

in degree $2(n-i+1)$ (cf. [D$_3$, Theorem 1]), and where $e_j(y_1^2,\cdots,y_n^2)$ is the j^{th} elementary symmetric function in the $y_1^2,\cdots,y_n^2$.

(iii) Bott-Samelson cycles on M can be obtained by restricting $h : \widetilde{M} \to M$ to appropriate subspaces of $\widetilde{M}$ (cf. Construction 2).

One can infer from (iii) the following result.

Theorem 7. *The induced ring map $h^* : H^*(M;\mathbb{Z}$ or $Z_2) \to H^*(\widetilde{M};\mathbb{Z}$ or $\mathbb{Z}_2)$ is injective. Furthermore*

(1) if $M = SO(n)$, then

$$h^*(\Omega(I)) = m_I(t_1,\cdots,t_{n'}),$$

where $m_I(t_1,\cdots,t_{n'})$ is the monomial symmetric function *in $t_1,\cdots,t_{n'}$ associated to the partition I ([D$_2$]);*

(2) if $M = G_{n,k}$, then

$$h^*(\Omega(I)) = S_I(x_1,\cdots,x_k),$$

where $S_I(x_1,\cdots,x_k)$ is the Schur Symmetric function *in $x_1,\cdots,x_k$ associated to the partition I ([D$_1$]);*

(3) if $M = CS_n$, then

$$h^*(\Omega(I)) = P_I(y_1,\cdots,y_n),$$

[2]The ring $H^*(G_2(\mathbb{R}^{2n});\mathbb{Z})$ is torsion free. The class $y^{n-1}+s$ is divisible by 2 because of $w_{2n-2}(\nu) \equiv s \equiv y^{n-1} \bmod 2$, where w_i is the i^{th} Stiefel-Whitney class.

where $P_I(y_1, \cdots, y_n)$ *is* the Schur P symmetric function *in* $y_1, \cdots, y_n$ *associated to the partition* I.□

(For definitions of these symmetric functions, see [Ma]).

Indeed, in each case concerned by Theorem 7, it can be shown that the $\Omega(I)$ are the Schubert classes [Ch, BGG].

It was first pointed out by Giambelli [G_1,G_2] in 1902 (see also Lesieur [L] or Tamvakis [T] in this volume) that multiplicative rule of Schubert classes in $G_{n,k}$ formally coincides with that of Schur functions, and by Pragacz in 1986 that multiplicative rule of Schubert classes in $\mathbb{C}S_n$ formally agree with that of Schur P functions [P, §6]. Many people asked why such similarities could possibly occur [S]. For instance it was said by C. Lenart [Le] that

> "No good explanation has been found yet for the occurrence of Schur functions in both the cohomology of Grassmanian and representation theory of symmetric groups".

Theorem 7 provides a direct linkage from Schubert classes to symmetric functions. It is for this reason combinatorial rules for multiplying symmetric functions of the indicated types (i.e. the *monomial symmetric functions, Schur symmetric functions* and *Schur P symmetric functions*) correspond to the intersection products of Schubert varieties in the spaces $M = SO(n)$, $G_{n,k}$ and $\mathbb{C}S_n$.

Remark. A link between representations and homogeneous spaces is furnished by Borel [B].

4–5. A concluding remark

Bott is famous for his periodicity theorem, which gives the homotopy groups of the matrix groups $O(n; \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ in the stable range. However, this part of Bott's work was improved and extended soon after its appearance [Ke], [HM], [AB].

It seems that the idea of Morse functions of Bott-Samelson type appearing nearly half century ago [BS_1, BS_2] deserves further attention. Recently, an analogue of Theorem 7 for the induced homomorphism

$$g_w^* : H^*(G/T) \to H^*(\Gamma_w)$$

of the Bott-Samelson cycle $g_w : \Gamma_w \to G/T$ (cf. Theorem 5) is obtained in [D_4, Lemma 5.1], from which the multiplicative rule of Schubert classes and the Steenrod operations on Schubert classes in a generalized flag manifold G/H [Ch, BGG] have been determined [D_4], [DZ_1], [DZ_2], where G is a compact connected Lie group, and where $H \subset G$ is the centralizer of a one-parameter subgroup in G.

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